# Velocity Defect Law for a Transpired Turbulent Boundary Layer

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RECENT experiments in our laboratory indicate that Clauser's[1] concept of the turbulent boundary layer formed on an impervious surface may be extended to the transpired turbulent boundary layer. The experimental equipment is described in detail elsewhere.[2] Two isothermal, air-to-air transpiration runs were performed, both at a constant mainstream velocity of 25 fps. The ratio of superficial injection velocity to mainstream velocity ($V_0/U_1$) was $1 \times 10^{-3}$ and $3 \times 10^{-3}$, respectively, for the two runs. The shear stress profiles were determined from the measured mean velocity profiles by hand computation employing momentum integral-type expressions. It was found that the shear stress rose rapidly from its wall value, soon reached a broad, well-defined maximum in the region $0.1 < y/\delta < 0.2$, and then decreased monotonically to zero at $y/\delta = 1$.

The velocity defect law applies only to the outer portion of the boundary layer, and a friction velocity based on the wall shear stress is normally employed as the scale velocity. Coles'[3,4] empirical correlation for a nontranspired boundary layer is

$$\frac{U_1 - U}{U_\tau} = -\frac{1}{K}\ln\frac{y}{\delta} + \frac{\pi(x)}{K}\left[2 - w\left(\frac{y}{\delta}\right)\right] \quad (1)$$

However, it is obvious from the preceding paragraph that the wall shear stress does not characterize adequately the processes occurring in the outer portion of the transpired boundary layer. Moreover, Clauser's model envisages an outer region of constant eddy viscosity that "floats" on a complicated substrate to which it is coupled only loosely. In view of these facts, it seems appropriate to employ a friction velocity $U_\tau{}^*$, based on the *maximum* shear stress, as the scale velocity. For flows with no transpiration and no axial pressure gradient, the maximum shear stress occurs at the wall, so this is perfectly consistent with current practice. Equation (1) then becomes

$$\frac{U_1 - U}{U_\tau{}^*} = -\frac{1}{K}\ln\frac{y}{\delta} + \frac{\pi(x)}{K}\left[2 - w\left(\frac{y}{\delta}\right)\right] \quad (2)$$

Furthermore, it is expected that the gross structure of the outer portion will be independent of transpiration rate. Hence, $K$, $\pi(x)$, and $w(y/\delta)$ should be unchanged by transpiration. The authors have employed the values most recently recommended by Coles[4] [$K = 0.41$ and $\pi(x) = 0.55$].

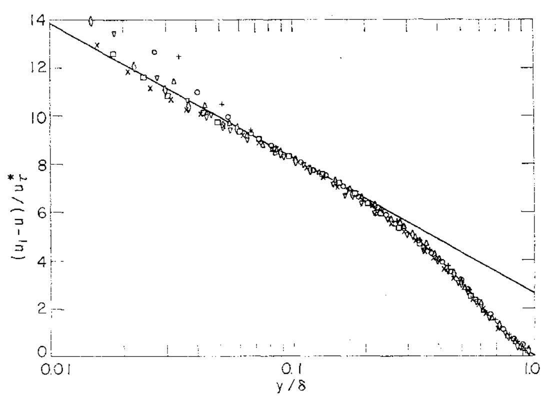


**Fig. 1** $V_0/U_1 = 1 \times 10^{-3}$

| | | | |
|---|---|---|---|
| + | $Re_\theta = 831$ | ▽ | $Re_\theta = 1506$ |
| ○ | $Re_\theta = 1047$ | ◊ | $Re_\theta = 1914$ |
| △ | $Re_\theta = 1293$ | □ | $Re_\theta = 2332$ |
| X | $Re_\theta = 2654$ | | |

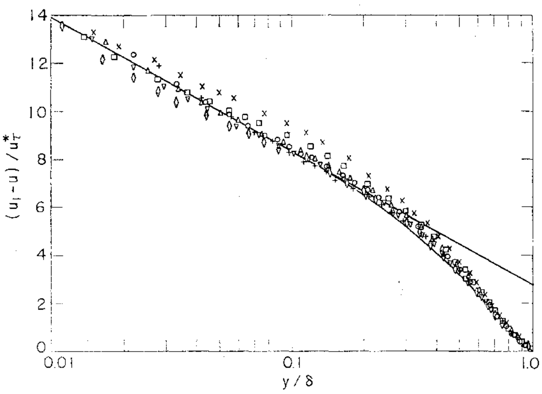


**Fig. 2** $V_0/U_1 = 3 \times 10^{-3}$

| | | | |
|---|---|---|---|
| + | $Re_\theta = 1170$ | ▽ | $Re_\theta = 2198$ |
| ○ | $Re_\theta = 1509$ | ◊ | $Re_\theta = 2980$ |
| △ | $Re_\theta = 2030$ | □ | $Re_\theta = 3701$ |
| X | $Re_\theta = 4411$ | | |

Figure 1 shows a comparison of Eq. (2) with the data obtained for $V_0/U_1 = 1 \times 10^{-3}$. Over the outer 90% of the boundary layer, the agreement is excellent. Figure 2 is the corresponding plot for $V_0/U_1 = 3 \times 10^{-3}$. In this case, the agreement is not quite as good, but it is well within the precision of the data.

These incomplete results indicate that Clauser's concept of a simple outer region "floating" on a complicated substrate is an extremely powerful one that may have real significance for transpired boundary layers. Currently the range and type of experimental measurements are being extended in order to examine the concept in more detail.

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# Simple Method of Analyzing Dissociative and Vibrational Relaxation behind Oblique Shock Waves

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RECENTLY, in several theoretical studies of dissociated boundary-layer flows with gas phase and/or catalytic surface reactions,[1–3] there has been exploited a remarkably simple yet accurate approximate method of predicting the nonequilibrium flow properties for arbitrary values of the reaction rate. The method consists of performing a local, nonlinear extrapolation of the exact first-order (nearly frozen)

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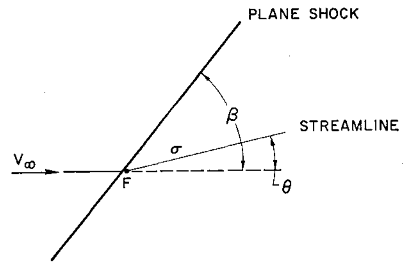


**Fig. 1 Flow configuration**

solution obtained for small values of the characteristic (flow time/reaction time) ratio Γ whereby the *form* of this solution is carried forward to arbitrary values of Γ. In view of the fact that this method has proven very useful in viscous flow applications, it becomes of interest to evaluate its potentialities in the case of inviscid nonequilibrium flows as well. Accordingly, this note describes the results of using the method to analyze nonequilibrium relaxation of a Lighthill ideal dissociating gas[4, 5] behind a strong, plane, oblique shock wave. The consequent implications concerning application to more complicated problems, such as dissociative or vibrational-relaxing flows around wedges and cones, also will be discussed.

## Analysis

Consider a relaxing, dissociated, one-dimensional flow of a diatomic gas behind an oblique shock wave as shown in Fig. 1. Epstein[6] has given exact solutions for the relaxing gas properties in the case of hypersonic flow using the Lighthill ideal gas model. Following his notation, the flow along a streamline can be described by the following three nondimensional equations for the atom mass fraction $\alpha$, density $\rho$, and pressure $p$, respectively:

$$\frac{d\alpha}{d\zeta} = F(\alpha,R,P) \equiv R^2\left[\frac{R(1+\alpha)}{P}\right]^S\left\{(1-\alpha)\times \exp\left[-\frac{DR(1+\alpha)}{2P}\right] - \frac{R}{R_D}\,\alpha^2\right\} \quad (1)$$

$$R \equiv \rho/\rho_\infty = (1 - P\csc^2\beta)^{-1} \quad (2)$$

$$P \equiv \frac{p}{\rho_\infty V_\infty^2} = \frac{3\sin^2\beta}{7+\alpha}\left\{1 + \left[1 + \frac{\alpha D}{9}\,(7 + 8\alpha + \alpha^2\csc^2\beta)\right]^{1/2}\right\} \quad (3)$$

where $\beta$ is the wave angle, $S$ is the recombination-rate temperature-dependence exponent, $R_D$ and $D$ are the nondimensional dissociation density and energy, respectively, and $\zeta \equiv C\mathcal{R}_u^S\rho_\infty\sigma/V_\infty^{2S+1}$ is the ratio of the distance $\sigma$ along the streamline to a characteristic dissociation relaxation distance. Note that the function $F$ defined in Eq. (1) has the important property of vanishing identically when equilibrium is reached behind the shock ($\zeta \to \infty$).

Now, near the shock where only a small degree of relaxation has occurred ($\zeta$ sufficiently small), Eqs. (1–3) may be seen to yield the following closed-form solutions:

$$\alpha - \alpha_F = \zeta F(\alpha_F,R_F,P_F) \quad (4)$$

$$R = R_F[1 + (\tfrac{7}{6}D\csc^2\beta - \tfrac{6}{7})\alpha] \quad (5)$$

$$P = P_F[1 + (\tfrac{7}{36}D\csc^2\beta - \tfrac{1}{7})\alpha] \quad (6)$$

where the subscript $F$ here denotes the chemically frozen conditions behind the shock front at $\zeta = 0$ (when $\alpha_F = 0$, $R_F = 7$, $P_F = \frac{6}{7}\sin^2\beta, \gamma_F = \frac{4}{3}$). These relations, of course, become progressively more inaccurate with increasing $\zeta$, as they do not account for the nonlinear reaction effects in Eq. (1). However, following the extrapolation concept discussed in Refs. 1–3, one may seek to carry this closed-form, nearly frozen solution forward to arbitrary values of $\zeta$ by assuming that Eqs. (4–6), which are rigorously correct only for $\zeta \ll 1$ are in fact nearly the correct form of the solution everywhere provided that the function $F$ is evaluated at the local values of $\alpha$, $R$, and $P$. Thus, the following is adopted tentatively as an approximate description of the relaxation process:

$$\alpha \simeq \zeta F[\alpha, R(\alpha), P(\alpha)] \quad (7)$$

with $R$ and $P$ still given by Eqs. (5) and (6). It may be noted that (7) becomes exact at sufficiently small $\zeta$, giving the correct initial values and gradients of all the dependent variables. Moreover, it is seen that (7) also gives the correct equilibrium relationship between $\alpha$, $R$, and $P$ in the asymptotic limit $\zeta \to \infty$.

Equations (5–7) yield an extremely simple, closed-form solution, since, by assuming values of $\alpha(\zeta)$, the corresponding location $\zeta$ easily is calculated from (7) for a given set of gas properties and shock angle. [The attendant values of $R$ and $P$ then can, if desired, be calculated directly from the exact relations (2) and (3).] A comparison of the dissociation distribution predicted by this approximate method with the exact solutions of Epstein for $\beta = 90°$ and $60°$ is shown in Fig. 2. It is seen that the nonlinear extrapolation of the nearly frozen solution gives a fairly accurate account of the nonequilibrium relaxation, being within 15% or less of the exact results for both wave angles. Moreover, the accuracy of the approximation appears to improve with decreasing shock strength, which is to be expected because of the correspondingly smaller nonlinear effects due to decreased initial dissociation rate and equilibrium degree of dissociation. It is concluded that the extrapolation described herein constitutes a very useful engineering tool for estimating the nonequilibrium flow properties in shocked diatomic gases. Furthermore, it is clear that the same idea may be applied to the case of vibrational energy relaxation as well.

## Concluding Remarks

The success of the extrapolation approximation illustrated in Fig. 2 for the simple case of plane shock wave flow suggests that this technique may find further application in dealing with the more complicated problem of vibrationally or dissociatively relaxing flow around slender aerodynamic bodies such as a wedge or cone. Here, for example, one proceeds by first obtaining a closed-form, nearly frozen solution along the body surface near the vertex, the initial gradients of the various flow properties along a streamline being evaluated as a function of the frozen flow solution from the relationships given by Sedney[7] and Hsu.[8] Then, one extrapolates nonlinearly the form of this solution to arbitrary distance along the body by evaluating the net reaction rate term locally as described previously. A preliminary analysis based on this approach, for the case of nonequilibrium vibration or dissociation in the hypersonic flow around a slender wedge, has been carried out with encouraging results; the results were found

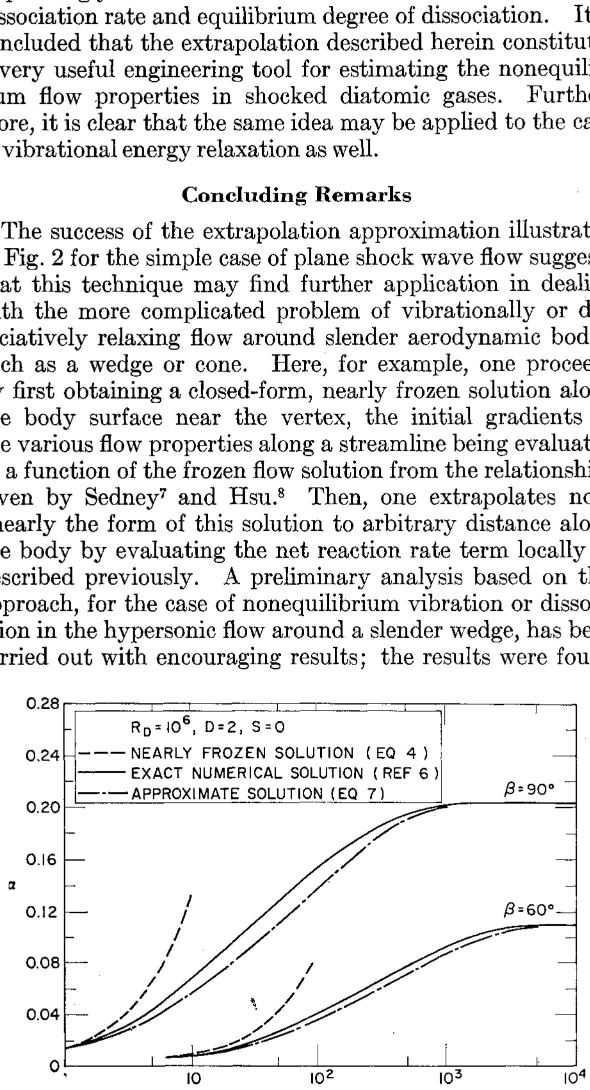


**Fig. 2 Comparison of exact and approximate solutions for atom distribution**

to predict the surface property distributions given by available exact numerical solutions[9, 10] within 15 to 25%. A more detailed discussion and evaluation of this work will be given in a future paper.


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# Effect of Surface Shear on Buckling of Cylindrical Shells


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In this note, a thin-walled circular cylindrical shell is assumed to be under surface shear loading in the longitudinal direction (Fig. 1). When the surface shear $\tau$ varies with $x$ only, the additional compression at one end of the cylinder is

$$P_1 = 2\pi R \int_0^l \tau(x)\, dx$$

By the principle of superposition, the shear $\tau$ can be considered as a combination of two parts. Referring to Fig. 2,

$$\tau = \tau_1 + \tau_2 \tag{1}$$

In the present case

$$\tau_1 = \tau_2 = \tau/2 \tag{2}$$

The role of $\tau_1$ can be considered that of a body-force component in $x$ direction. Hence, the equilibrium conditions in $x$ and $y$ directions are, respectively,

$$\begin{aligned}(\partial\sigma_x/\partial x) + (\partial\sigma_{xy}/\partial y) + (2\tau_1/t) &= 0\\ (\partial\sigma_y/\partial y) + (\partial\sigma_{xy}/\partial x) &= 0\end{aligned} \tag{3}$$

The potential function $V$ is introduced such that

$$\begin{aligned}\partial V/\partial x &= -(\tau/t)\\ \partial V/\partial y &= 0\end{aligned} \tag{4}$$

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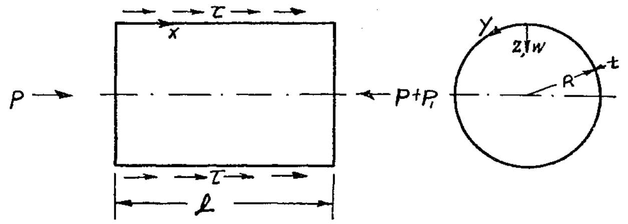


**Fig. 1 Cylinder under longitudinal shear and axial compression**

When Eqs. (4) are substituted into Eqs. (3) and the terms due to large deflection in the radial direction are included, the compatibility equation has the following form:

$$\nabla^2(\sigma_x + \sigma_y) = (1 + \nu)\nabla^2 V + f(w) \tag{5}$$

In Eq. (5)

$$f(w) = E\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right) - \frac{1}{R}\frac{\partial^2 w}{\partial x^2}\right] \tag{6}$$

where $\nu$ is the Poisson ratio, $w$ the radial deflection, and $\nabla^2$ the Laplacian operator.

The stress function $\varphi(x,y)$ is defined by

$$\begin{aligned}\sigma_x - V &= \partial^2\varphi/\partial y^2\\ \sigma_y - V &= \partial^2\varphi/\partial x^2\\ \sigma_{xy} &= -(\partial^2\varphi/\partial x\partial y)\end{aligned} \tag{7}$$

From Eqs. (7) and (5), the compatibility equation becomes

$$\nabla^4\varphi = -(1 - \nu)\nabla^2 V + E\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) - \frac{1}{R}\frac{\partial^2 w}{\partial x^2}\right] \tag{8}$$

The equilibrium condition in the radial direction and the equilibrium relations of moments are found by modifying those given in Ref. 1. These relations are

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + t\left[\sigma_y\left(\frac{1}{R} + \frac{\partial^2 w}{\partial y^2}\right) + \sigma_x\frac{\partial^2 w}{\partial x^2} + 2\sigma_{xy}\frac{\partial^2 w}{\partial x\partial y}\right] - 2\tau_1\frac{\partial w}{\partial x} = 0 \tag{9}$$

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} - Q_x - \tau_2 t = 0 \tag{10}$$

$$\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} - Q_y = 0 \tag{11}$$

From Eqs. (9–11), the equilibrium equation can be expressed as

$$D\nabla^4 w = t\left[\sigma_y\left(\frac{1}{R} + \frac{\partial^2 w}{\partial y^2}\right) + \sigma_x\frac{\partial^2 w}{\partial x^2} + 2\sigma_{xy}\frac{\partial^2 w}{\partial x\partial y}\right] + \frac{t^2}{2}\nabla^2 V + t\frac{\partial V}{\partial x}\frac{\partial w}{\partial x} \tag{12}$$

The solution can be found from coupling Eq. (8) with Eq. (12). It should be noted that these equations are analogous

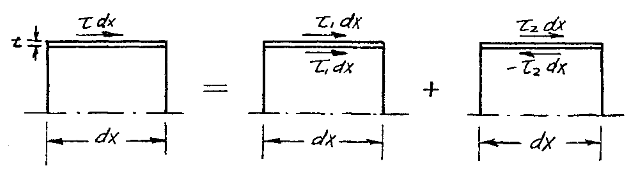


**Fig. 2 Equivalence of shear forces ($t$ = thickness, $R$ = radius of cylinder)**